PG – 204

Second Semester M.Sc. Degree Examination, July 2017
(CBCS Scheme)
MATHEMATICS
M 201T : Algebra – II

Time : 3 Hours Max. Marks : 70

Instructions : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Define a nilradical of a commutative ring. Show that the nilradical N(A) of the ring A is equal to the intersection of all prime.

b) Let $f : A \to B$ be a ring homomorphism. Let I and J be ideals of A and B, respectively. Then show that

i) $I \subseteq I^{ec}$, $J \supseteq J^{ce}$

ii) $I^e = I^{ece}$, $J^c = J^{cec}$

iii) If C denotes the set of all contacted ideals of A and E denotes the set of all extended ideals of B, then $C = \{I : I^{ec} = I\}$ and $E = \{J : J^{ce} = J\}$. Further, show that the map $I \to I^e$ is bijective map of C onto E, whose inverse is $J \to J^c$. **(6+8)**

2. a) If $L \supseteq M \supseteq N$ are A-modules, then show that $\left(L/N\right) \left(M/N\right) \cong L/M$.

b) Define a finitely generated free A-module. Prove that sub module of finitely generated modules need not be finitely generated.

c) State and prove the Nakayama Lemma. **(4+4+6)**

3. a) Let M and N be simple A-modules. Then prove that A-linear map $f : M \to N$ is either zero or an isomorphism. Further, show that the ring of endomorphism of M is a division ring.

b) Define an exact sequence of an A-module. Show that the sequence of A-module and A-linear maps $M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ is exact, if for all A-modules N, the sequence $0 \to \mathrm{Hom}(M'', N) \xrightarrow{\bar{v}} \mathrm{Hom}(M, N) \xrightarrow{\bar{u}} \mathrm{Hom}(M', N)$ is exact. **(7+7)**

P.T.O.

4. a) Define a Noetherian module. Let N be a sub module of M. Then show that M is Noetherian if and only if N and M/N are Noetherian.

   b) Show that a commutative ring with identity is Noetherian if and only if strictly ascending chain of ideals is of finite length.

   c) Show that an Artinian integral domain is a field. **(6+5+3)**

5. a) Define an algebraic extension of a field. If L is an algebraic extension of K and K is an algebraic extension of F, then prove that L is an algebraic extension of F.

   b) Prove that the elements in an extension K of a field F which are algebraic over F form a subfield of K.

   c) Let $a = \sqrt{2}, b = \sqrt[4]{2}$, where R is an extension of Q. Verify that (a + b) and (ab) are algebraic of degree atmost (dega) (degb). **(5+5+4)**

6. a) State and prove the Remainder and Factor theorem for root of a polynomial.

   b) Let $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$. Then prove that three is an extension E of F of degree atmost n! in which f(x) has n-roots.

   c) Determine the splitting field over rationals of

      i) Quadratic polynomial

      ii) Cubic polynomial. **(4+6+4)**

7. a) Prove that it is impossible to trisect 60° by using straight edge and compass.

   b) Show that the polynomial $f(x) \in F[x]$ has multiple roots if and only if f(x) and $f'(x)$ have a non-trivial common factors.

   c) Define a simple extension of a field with an example. Show that any finite extension of a field of characteristic zero is a simple extension. **(4+6+4)**

8. a) Define a fixed field. Show that the fixed field is a subfield of a field K.

   b) Define a normal extension of a field. Show that K is a normal extension of a field F of characteristic 0 if and only if K is a splitting field of same polynomial over F.

   c) State fundamental theorem of Galois theory. **(4+8+2)**